Group Theory 2

22 September 2023 09:59

$\sigma = (1\ 2)(3\ 4)(5\ 6)(7\ 8)(9\ 10)$ S_{10} τ^k

$\tau^5 = \sigma$

$\boxed{\frac{n}{\gcd(n,k)} = 2} \Rightarrow \frac{10}{\gcd(n,k)} = 2 \Rightarrow \gcd(n,k) = 5 \Rightarrow k = 5$ as $n = 10$

$\tau = (\underline{1}\ \underline{3}\ \underline{5}\ \underline{7}\ \underline{9}\ \underline{2}\ \underline{4}\ \underline{6}\ \underline{8}\ \underline{10})$

$\sigma = (a_1\, a_2 \ldots a_{k_1})(a_{k_1+1} \ldots a_{k_2}) \ldots (a_{k_{n-1}+1} \ldots a_{k_n})$

Q) If $\alpha, \beta \in S_n$ are disjoint and $\alpha\beta = 1$ then show that $\alpha = \beta = 1$

Ans:- $\alpha(\beta(x)) = x \Rightarrow \alpha(x) \neq x \Rightarrow \beta(x) = x$

$\beta(y) \neq y \Rightarrow \alpha(y) = y$

$\exists x$ such that $\beta(x) \neq x$

$\Rightarrow \alpha(\beta(x)) = \beta(x) \neq x$

But we know $\alpha\beta = 1 \Rightarrow \Leftarrow$

Thus $\beta(x) = x\ \forall x \in S_x$

Similarly $\alpha(x) = x\ \forall x \in S_x$

$\Rightarrow \alpha = \beta = 1$

Q) Let $\alpha = \beta\gamma$ in S_n where β and γ are disjoint. Show that if β moves i then $\alpha^k(i) = \beta^k(i)\ \forall\, k \geq 0$

Ans:- $\alpha(i) = \beta\gamma(i) = \beta(\gamma(i)) = \beta(i)$

$\Rightarrow \alpha^k(i) = \beta^k(i)$ ... [using induction]

Q) If α and β are cycles in S_n and if $\exists\ i$ moved by both α and β and if $\alpha^k(i) = \beta^k(i)\ \forall k \in \mathbb{Z}$, then show that $\alpha = \beta$

Ans:- $\alpha = \beta\gamma$

$\alpha = (\beta_1\, \beta_2 \ldots \beta_r)(\gamma_1\, \gamma_2 \ldots \gamma_k)$ $\gamma^k(1\ 2 \quad r+k) = (1\ 2 \ldots r+k)$

$\alpha^k(1\ 2 \ldots r+k) = \beta^k(1\ 2 \ldots r+k)$ $\gamma = 1$

$\Rightarrow \alpha = \beta$

Factorizations into Disjoint Cycles :-

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 4 & 1 & 2 & 5 & 3 & 8 & 9 & 7 \end{pmatrix}$$

$$\alpha = (1\ 6\ 3)(2\ 4)(5)(7\ 8\ 9) = (1\ 6\ 3)(2\ 4)(7\ 8\ 9)$$

Theorem :- Every permutation $\alpha \in S_n$ is either a cycle or a disjoint product of cycle

Definition :- A complete factorization of a permutation α is a factorization of α as a product of disjoint cycles which contains one 1-cycle (i) for every i fixed by α.

Theorem :- Let $\alpha \in S_n$ and let $\alpha = \beta_1 \beta_2 \cdots \beta_t$ be a complete factorization into disjoint cycles. This factorization is unique for the order in which the factor occurs

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Q> α is a permutation of $\{1, 2, \ldots, 9\}$ defined by $\alpha(i) = 10 - i$. Write α as a product of disjoint cycles

Ans :- $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$

$= (1\ 9)(2\ 8)(3\ 7)(4\ 6)$

Q> How many $\alpha \in S_n$ are there with $\alpha^2 = 1$?

Ans :- $(i\ j)\ \forall\ i, j \in X \rightarrow \binom{n}{2}$

$(i\ j)(r\ s)\ \forall\ i, j, r, s \in X \rightarrow \binom{n}{2}\binom{n-2}{2}$

$\Sigma\left(\prod_{i=0}^{n/2} \binom{n-2i}{2}\right)$

Even and Odd Permutations :-

Theorem :- Every permutation $\alpha \in S_n$ is a product of transpositions

Proof :- $(a_1 a_2 \cdots a_k) = (a_1 \; a_k)(a_1 \, a_{k-1}) \cdots (a_1 \, a_2)$

Definition :- A permutation $\alpha \in S$ is <u>even</u> if it is a product of an even number of transpositions otherwise it is <u>odd</u>.

	Parity
$1 \; 2 \; 3 \; 4 \; 5 \; 6 \; 7 \; 8 \longrightarrow$	0
$1 \; 3 \; 4 \; 5 \; 6 \; 7 \; 8 \; 2 \longrightarrow$	0
$1 \; 3 \; 2 \; 4 \; 5 \; 6 \; 7 \; 8 \longrightarrow$	1
$1 \; 3 \; 2 \; 4 \; 5 \; 6 \; 8 \; 7 \longrightarrow$	0

Lemma :- If $k, l \geq 0$ then

$$(a \; b)(a \; c_1 \cdots c_k \; b \; d_1 \cdots d_l) = (a \; c_1 \cdots c_k)(b \; d_1 \cdots d_l)$$

and

$$(a \; b)(a \; c_1 \cdots c_k)(b \; d_1 \cdots d_l) = (a \; c_1 \cdots c_k \; b \; d_1 \cdots d_l)$$